Chirality dynamics for a fermionic particle non-minimally coupling with an external magnetic field

A E Bernardini

Instituto de Física Gleb Wataghin, UNICAMP, PO Box 6165, SP 13083-970, Campinas, SP, Brazil

E-mail: alexeb@ifi.unicamp.br

Abstract
We proceed with the construction of normalizable Dirac wave packets for treating chiral oscillations in the presence of an external magnetic field. Both chirality and helicity quantum numbers correspond to variables of fundamental importance in the study of chiral interactions, in particular, in the context of neutrino physics. In order to clarify a subtle aspect in the confront of such concepts which, for massive particles, represent different physical quantities, we are specifically interested in quantifying chiral oscillations for a *fermionic* Dirac-*type* particle (neutrino) non-minimally coupling with an external magnetic field B by solving the corresponding interacting Hamiltonian (Dirac) equation. The viability of the intermediate wave-packet treatment becomes clear when we assume B orthogonal/parallel to the direction of the propagating particle.

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Since those early days when Dirac derived the relativistic wave equation for a free propagating electron [1], several efforts have been produced in the literature to solve the Dirac equation with other analytical forms of interacting potentials, from central potential solutions [2, 3] to recent theoretical attempts to describe quark confinement [4, 5]. In fact, obtaining exact solutions of a generic class of Dirac wave equations becomes important since, for many times, the conceptual understanding of physics can only be brought about by such solutions. These solutions also correspond to valuable means for checking and improving models and numerical methods for solving complicated physical problems. In the context in which we intend to explore the Dirac formalism, we can take into account the effect of neutrino spin flipping attributed to some dynamic external [6] interacting processes which come from the

non-minimal coupling of a magnetic moment with an external electromagnetic field [7] and which was formerly supposed to be a relevant effect in the context of the solar-neutrino puzzle by suggesting an explanation for the LSND anomaly [8–10][1].

In the corresponding theoretical framework, we can note a more extended discussion involving the evolution equation of the covariant neutrino spin operator in the Heisenberg representation in the presence of general external fields [16]. Such an analysis involves both electromagnetic and weak interactions with background matter [16, 17]. In a similar context, some developments considering the minimal coupling of an electrically charged particle with an external magnetic field were also performed for an electron described by the Dirac equation [18] and the theory of spin light of neutrino in matter and electromagnetic fields has been extensively studied [19].

We know, however, that independently of any external electromagnetic field, since neutrinos are detected essentially via V–A charged weak currents, the chiral oscillation mechanism by itself may even provide a solution to the posed anomaly. The point is that an erroneous correspondence is supposed to cause some confusion in the literature where the concept of helicity has been currently used in the place of chirality. In the standard model of flavour-changing interactions, neutrinos with positive chirality are decoupled from neutrino absorbing charged weak currents [24]. Consequently, neutrinos with positive chirality are sterile with respect to weak interactions. By reporting to the formalism with Dirac wave packets [25, 26], which leads to the study of chiral oscillations [24] (in vacuum), we are now interested in appointing the modifications of the chirality dynamics which are observed when the neutrino electrodynamics is accurately discussed.

In this context, our aim in this paper is to try to accommodate the Dirac wave-packet formalism [25, 26] and a further class of static characteristics of neutrinos, namely, the (electro)magnetic moment which appears in a Lagrangian with non-minimal coupling. By following this line of reasoning, we are not only interested in solving a 'modified' Dirac wave equation but also in constructing Dirac wave packets which can be used for describing the dynamics of chirality, a physical variable which is relevant in the context of the quantum oscillation phenomena.

Despite their electric charge neutrality, neutrinos can interact with a photon through loop (radiative) diagrams. The Lagrangian for the interaction between a fermionic field $\psi(x)$ and an electromagnetic field written in terms of the field-strength tensor $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ is given by

$$\mathcal{L} = \tfrac{1}{2}\overline{\psi}(x)\sigma_{\mu\nu}[\mu F^{\mu\nu}(x) - d\mathcal{F}^{\mu\nu}(x)]\psi(x) + \text{h.c.}, \tag{1}$$

where $x = (t, \boldsymbol{x})$, $\sigma_{\mu\nu} = \tfrac{i}{2}[\gamma_\mu, \gamma_\nu]$, the *dual* field-strength tensor $\mathcal{F}^{\mu\nu}(x)$ is given by $\mathcal{F}^{\mu\nu}(x) = \tfrac{1}{2}\epsilon^{\mu\nu\lambda\delta}F^{\lambda\delta}(x)$ [2] and the coefficients μ and d represent, respectively, the magnetic and the electric dipole moment which establish the neutrino electromagnetic coupling. One can note that we have not discriminated the flavour/mass mixing elements in the above interacting Lagrangian since we are indeed interested in the physical observable dynamics ruled by the Hamiltonian

$$\begin{aligned} H &= \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m - \beta\left[\frac{\sigma_{\mu\nu}}{2}(\mu F^{\mu\nu}(x) - d\mathcal{F}^{\mu\nu}(x)) + \text{h.c.}\right] \\ &= \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta[m - \mu\boldsymbol{\Sigma} \cdot \boldsymbol{B}(x) + d\boldsymbol{\Sigma} \cdot \boldsymbol{E}(x)], \end{aligned} \tag{2}$$

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[1] In fact, the advent of the neutrino physics [11–13] has stirred up an increasing number of theoretical works where the quantum oscillation mechanisms [14, 15, 21–23] have been deeply analysed. Although clear experimental evidence is still missing, it is widely believed that neutrino mixing and oscillations are the basic tool for further understanding of neutrino phenomenology.

[2] $\epsilon^{\mu\nu\lambda\delta}$ is the totally fourth rank antisymmetric tensor.

where, in terms of the Dirac matrices, $\boldsymbol{\alpha} = \sum_{k=1}^{3} \alpha_k \hat{k} = \sum_{k=1}^{3} \gamma_0 \gamma_k \hat{k}$, $\beta = \gamma_0$ and $\boldsymbol{B}(x)$ and $\boldsymbol{E}(x)$ are, respectively, the magnetic and electric fields. In fact, equation (2) could be extended to an equivalent matrix representation with flavour and mass mixing elements where the diagonal (off-diagonal) elements described by $\mu_{i,j}(m_{i,j})$ and $d_{i,j}(m_{i,j})$, where i, j are mass indices, would be called diagonal (transition) moments. In this context, for both Dirac and Majorana neutrinos, we could have transition amplitudes with non-vanishing magnetic and electric dipole moments [27–29]. Otherwise, the CP invariance holds the diagonal electric dipole moments null [30]. Specifically for Majorana neutrinos, it can be demonstrated that the diagonal magnetic and electric dipole moments vanish if CPT invariance is assumed [27].

Turning back to the simplifying example of diagonal moments and assuming CP and CPT invariance, we can restrict our analysis to the coupling with only an external magnetic field $\boldsymbol{B}(x)$ by setting $d = 0$. From this point, the expression for μ can be found from Feynman diagrams for magnetic moment corrections [30] and turns out to be proportional to the neutrino mass (matrix),

$$\mu = \frac{3eG}{8\sqrt{2}\pi^2} m = \frac{3m_e G}{4\sqrt{2}\pi^2} \mu_B m_\nu = 2.7 \times 10^{-10} \mu_B \frac{m_\nu}{m_N}, \tag{3}$$

where G is the Fermi constant and m_N is the nucleon mass[3]. In particular, for $m_\nu \approx 1$ eV, the magnetic moment introduced by the above formula is exceedingly small to be detected or to affect astrophysical or physical processes.

Since we are interested in constructing the dynamics ruled by the Hamiltonian of equation (2), we first observe that the free propagating momentum is not a conserved quantity,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{p} \rangle = \mathrm{i} \langle [H, \boldsymbol{p}] \rangle = \mathrm{Re}(\mu) \langle \beta \boldsymbol{\nabla}(\boldsymbol{\Sigma} \cdot \boldsymbol{B}(x)) \rangle. \tag{4}$$

In the same way, the particle velocity given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{x} \rangle = \mathrm{i} \langle [H, \boldsymbol{x}] \rangle = \langle \boldsymbol{\alpha} \rangle \tag{5}$$

comes out as a non-null value. After solving the $\boldsymbol{x}(t)(\boldsymbol{\alpha}(t))$ differential equation, it is possible to observe that, in addition to a uniform motion, the fermionic particle executes very rapid oscillations known as *zitterbewegung* [32]. By following an analogous procedure for the Dirac chiral operator γ_5, newly recurring to the equation of the motion, it is possible to have the chirality and the helicity dynamics respectively given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \gamma^5 \rangle = 2\mathrm{i} \langle \gamma_0 \gamma_5 [m - \mu \boldsymbol{\Sigma} \cdot \boldsymbol{B}(x)] \rangle \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle h \rangle = \frac{1}{2} \mu \langle \gamma_0 [(\boldsymbol{\Sigma} \cdot \boldsymbol{\nabla})(\boldsymbol{\Sigma} \cdot \boldsymbol{B}(x)) + 2(\boldsymbol{\Sigma} \times \boldsymbol{B}(x)) \cdot \boldsymbol{p}] \rangle, \tag{7}$$

where we have alternatively defined the particle helicity as the projection of the spin angular momentum onto the vector momentum, $h = \frac{1}{2} \boldsymbol{\Sigma} \cdot \boldsymbol{p}$ (with $\boldsymbol{p}$ in place of $\hat{\boldsymbol{p}}$). From equations (6)–(7) we can state that if a neutrino has an intrinsic magnetic moment and passes through a region filled by a magnetic field, the neutrino helicity can flip in a completely different way from how chiral oscillations evolve in time. In the non-interacting case, it is possible to verify that the time-dependent averaged value of the Dirac chiral operator γ_5 has an oscillating behaviour [24] very similar to the rapid oscillations of the position. Equations (6)–(7) can be reduced to the non-interacting case by setting $\boldsymbol{B}(x) = 0$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle h \rangle = \mathrm{i} \langle [H, h] \rangle = -\langle (\boldsymbol{\alpha} \times \boldsymbol{p}) \cdot \hat{\boldsymbol{p}} \rangle = 0 \tag{8}$$

---

[3] We are using some results of the standard $SU(2)_L \otimes U(1)_Y$ electroweak theory [31].

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma^5\rangle = \mathrm{i}\langle[H,\gamma^5]\rangle = 2\mathrm{i}m\langle\gamma^0\gamma^5\rangle, \tag{9}$$

from which we confirm that the chiral operator γ^5 is *not* a constant of the motion [24]. The effective value of equation (9) appears only when both positive and negative frequencies are taken into account to compose a Dirac wave packet, i.e. the non-null expectation value of $\langle\gamma_0\gamma_5\rangle$ is revealed by the interference between Dirac equation solutions with opposite sign frequencies. The effective contribution due to this interference effect led us to report the Dirac wave-packet formalism in order to quantify neutrino chiral oscillations in the presence of an external magnetic field.

Assuming the simplifying hypothesis of a uniform magnetic field $\boldsymbol{B}$, the physical implications of the non-minimal coupling with an external magnetic field can then be studied by means of the eigenvalue problem expressed by the Hamiltonian equation

$$H(\boldsymbol{p})\varphi_n = E_n(\boldsymbol{p})\varphi_n = \{\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta[m - \mu\boldsymbol{\Sigma}\cdot\boldsymbol{B}]\}\varphi_n, \tag{10}$$

for which the explicit 4×4 matrix representation is given by

$$H(\boldsymbol{p})\varphi_n = \begin{bmatrix} m - \mu B_z & -\mu(B_x - \mathrm{i}B_y) & p_z & p_x - \mathrm{i}p_y \\ -\mu(B_x + \mathrm{i}B_y) & m + \mu B_z & p_x + \mathrm{i}p_y & -p_z \\ p_z & p_x - \mathrm{i}p_y & -(m - \mu B_z) & \mu(B_x - \mathrm{i}B_y) \\ p_x + \mathrm{i}p_y & -p_z & \mu(B_x + \mathrm{i}B_y) & -(m + \mu B_z) \end{bmatrix}\varphi_n. \tag{11}$$

The most general eigenvalue ($E_n(\boldsymbol{p})$) solution of the above problem is given by

$$E_n(\boldsymbol{p}) = \pm E_s(\boldsymbol{p}) = \pm\sqrt{m^2 + \boldsymbol{p}^2 + \boldsymbol{a}^2 + (-1)^s 2\sqrt{m^2\boldsymbol{a}^2 + (\boldsymbol{p}\times\boldsymbol{a})^2}}, \qquad s = 1, 2, \tag{12}$$

where we have denoted $E_{n=1,2,3,4} = \pm E_{s=1,2}$ and set $\boldsymbol{a} = \mu\boldsymbol{B}$. The complete set of orthonormal eigenstates φ_n thus can be written in terms of the eigenfunctions $\mathcal{U}(p_s)$ with positive energy eigenvalues ($+E_s(\boldsymbol{p})$) and the eigenfunctions $\mathcal{V}(p_s)$ with negative energy eigenvalues ($-E_s(\boldsymbol{p})$),

$$\mathcal{U}(p_s) = -N(p_s)\left\{\sqrt{\frac{A_s^-}{A_s^+}}, \sqrt{\frac{\alpha_s^+}{\alpha_s^-}}, \sqrt{\frac{A_s^-\alpha_s^+}{A_s^+\alpha_s^-}}, -1\right\}^\dagger$$
$$\mathcal{V}(p_s) = -N(p_s)\left\{\sqrt{\frac{A_s^-}{A_s^+}}, -\sqrt{\frac{\alpha_s^-}{\alpha_s^+}}, -\sqrt{\frac{A_s^-\alpha_s^-}{A_s^+\alpha_s^+}}, -1\right\}^\dagger, \tag{13}$$

where p_s is the relativistic *quadrimomentum*, $p_s = (E_s(\boldsymbol{p}), \boldsymbol{p})$, $N(p_s)$ is the normalization constant and

$$A_s^\pm = \Delta_s^2(\boldsymbol{p}) \pm 2m|\boldsymbol{a}| - \boldsymbol{a}^2, \qquad \alpha_s^\pm = 2E_s(\boldsymbol{p})|\boldsymbol{a}| \pm \left(\Delta_s^2(\boldsymbol{p}) + \boldsymbol{a}^2\right), \tag{14}$$

with

$$\Delta_s^2(\boldsymbol{p}) = E_s^2(\boldsymbol{p}) - (m^2 + \boldsymbol{p}^2) + \boldsymbol{a}^2. \tag{15}$$

We can observe that the above spinorial solutions are free of any additional constraint, namely, at a given time t, they are independent functions of $\boldsymbol{p}$ and they do not represent chirality/helicity eigenstates.

In order to describe the above Hamiltonian dynamics for a generic observable $\mathcal{O}(t)$ we can first seek a generic plane-wave decomposition as

$$\begin{aligned} &\exp[-\mathrm{i}(E_s(\boldsymbol{p})t - \boldsymbol{p}\cdot\boldsymbol{x})]\mathcal{U}(p_s), && \text{for positive frequencies and} \\ &\exp[\mathrm{i}(E_s(\boldsymbol{p})t - \boldsymbol{p}\cdot\boldsymbol{x})]\mathcal{V}(p_s), && \text{for negative frequencies,} \end{aligned} \tag{16}$$

so that the time evolution of a plane wave packet $\psi(t, \boldsymbol{x})$ can be written as

$$
\begin{aligned}
\psi(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \sum_{s=1,2} \{ & b(p_s)\mathcal{U}(p_s) \exp[-iE_s(\boldsymbol{p})t] \\
& + d^*(\tilde{p}_s)\mathcal{V}(\tilde{p}_s) \exp[+iE_s(\boldsymbol{p})t] \} \exp[i\boldsymbol{p} \cdot \boldsymbol{x}],
\end{aligned}
\tag{17}
$$

with $\tilde{p}_s = (E_s, -\boldsymbol{p})$. Meanwhile, equation (17) requires some extensive mathematical manipulations for explicitly constructing the dynamics of an operator $\mathcal{O}(t)$ like

$$
\mathcal{O}(t) = \int \mathrm{d}^3 \boldsymbol{x} \, \psi^\dagger(t, \boldsymbol{x}) \mathcal{O} \psi(t, \boldsymbol{x}).
\tag{18}
$$

If, however, the quoted observables like the chirality γ^5, the helicity h or even the spin projection onto $\boldsymbol{B}$ commuted with the Hamiltonian H, we could reconfigure the above solutions to the simpler ones. To illustrate this point, let us limit our analysis to very restrictive spatial configurations of $\boldsymbol{B}$ so that, as a first attempt, we can calculate the observable expectation values which appear in equation (6). Let us then assume that the magnetic field $\boldsymbol{B}$ is either orthogonal or parallel to the momentum $\boldsymbol{p}$. For both of these cases the spinor eigenstates can then be decomposed into orthonormal bi-spinors as

$$
\mathcal{U}(p_s) = N^+(p_s) \begin{bmatrix} \varphi^+(p_s) \\ \chi^+(p_s) \end{bmatrix}
\tag{19}
$$

and

$$
\mathcal{V}(p_s) = N^-(p_s) \begin{bmatrix} \varphi^-(p_s) \\ \chi^-(p_s) \end{bmatrix}.
\tag{20}
$$

Eventually, in order to simplify some subsequent calculations involving chiral oscillations, we could set $\varphi_{1,2}^\pm$ and $\chi_{1,2}^\pm$ as eigenstates of the spin projection operator $\boldsymbol{\sigma} \cdot \boldsymbol{B}$, i.e. besides being energy eigenstates, the general solutions $\mathcal{U}(p_s)$ and $\mathcal{V}(p_s)$ would become eigenstates of the operator $\boldsymbol{\Sigma} \cdot \boldsymbol{B}$ and, equivalently, of $\boldsymbol{\Sigma} \cdot \boldsymbol{a}$.

Now equation (10) can be decomposed into a pair of coupled equations like

$$
(\pm E_s - m + \boldsymbol{\sigma} \cdot \boldsymbol{a})\varphi_s^\pm = \pm \boldsymbol{\sigma} \cdot \boldsymbol{p} \chi_s^\pm, \qquad (\pm E_s + m - \boldsymbol{\sigma} \cdot \boldsymbol{a})\chi_s^\pm = \pm \boldsymbol{\sigma} \cdot \boldsymbol{p} \varphi_s^\pm,
\tag{21}
$$

where we have suppressed the p_s dependence. By introducing the commuting relation $[\boldsymbol{\sigma} \cdot \boldsymbol{p}, \boldsymbol{\sigma} \cdot \boldsymbol{B}] = 0$, which is derived when $\boldsymbol{p} \times \boldsymbol{B} = 0$, the eigenspinor representation can be reduced to

$$
\mathcal{U}(p_s) = \sqrt{\frac{E_s + m_s}{2E_s}} \begin{bmatrix} \varphi_s^+ \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E_s + m_s} \varphi_s^+ \end{bmatrix}
\tag{22}
$$

and

$$
\mathcal{V}(p_s) = \sqrt{\frac{E_s + m_s}{2E_s}} \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E_s + m_s} \chi_s^- \\ \chi_s^- \end{bmatrix},
\tag{23}
$$

with $m_s = m - (-1)^s |a|$ and the energy eigenvalues

$$
\pm E_s = \pm\sqrt{\boldsymbol{p}^2 + m_s^2}.
\tag{24}
$$

In this case, the closure relations can be constructed in terms of

$$
\begin{aligned}
\sum_{s=1,2} \mathcal{U}(p_s) \otimes \mathcal{U}^\dagger(p_s)\gamma_0 &= \sum_{s=1,2} \left\{ \frac{\gamma_\mu p_s^\mu + m_s}{2E_s} \left[\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right] \right\} \\
-\sum_{s=1,2} \mathcal{V}(p_s) \otimes \mathcal{V}^\dagger(p_s)\gamma_0 &= \sum_{s=1,2} \left\{ \frac{-\gamma_\mu p_s^\mu + m_s}{2E_s} \left[\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right] \right\}.
\end{aligned}
\tag{25}
$$

Analogously, by introducing the anti-commuting relation $\{\boldsymbol{\sigma} \cdot \boldsymbol{p}, \boldsymbol{\sigma} \cdot \boldsymbol{B}\}$ when $\boldsymbol{p} \cdot \boldsymbol{B} = 0$, the eigenspinor representation can be reduced to

$$\mathcal{U}(p_s) = \sqrt{\frac{\varepsilon_0 + m}{2\varepsilon_0}} \begin{bmatrix} \varphi_s^+ \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{\varepsilon_0 + m} \varphi_s^+ \end{bmatrix} \tag{26}$$

and

$$\mathcal{V}(p_s) = \sqrt{\frac{\varepsilon_0 + m}{2\varepsilon_0}} \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{\varepsilon_0 + m} \chi_s^- \\ \chi_s^- \end{bmatrix}, \tag{27}$$

with $\varepsilon_0 = \sqrt{\boldsymbol{p}^2 + m^2}$ and the energy eigenvalues

$$\pm E_s = \pm[\varepsilon_0 + (-1)^s |\boldsymbol{a}|]. \tag{28}$$

In this case, the closure relations can be constructed in terms of

$$\begin{aligned} \sum_{s=1,2} \mathcal{U}(p_s) \otimes \mathcal{U}^\dagger(p_s)\gamma_0 &= \frac{\gamma_\mu p_0^\mu + m}{2\varepsilon_0} \sum_{s=1,2} \left[\frac{1 - (-1)^s \gamma_0 \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right] \\ - \sum_{s=1,2} \mathcal{V}(p_s) \otimes \mathcal{V}^\dagger(p_s)\gamma_0 &= \frac{-\gamma_\mu p_0^\mu + m}{2\varepsilon_0} \sum_{s=1,2} \left[\frac{1 - (-1)^s \gamma_0 \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right], \end{aligned} \tag{29}$$

where $p_0 = (\varepsilon_0, \boldsymbol{p})$.

Since we can set $\varphi_{1,2}^+ \equiv \chi_{1,2}^-$ as the components of an orthonormal basis, we can immediately deduce the orthogonality relations

$$\mathcal{U}^\dagger(p_s)\mathcal{U}(p_r) = \mathcal{V}^\dagger(p_s)\mathcal{V}(p_r) = \delta_{sr}, \qquad \mathcal{U}^\dagger(p_s)\gamma_0\mathcal{V}(p_r) = \mathcal{V}^\dagger(p_s)\gamma_0\mathcal{U}(p_r) = 0, \tag{30}$$

which are valid for both of the above cases.

Finally, the calculation of the expectation value of $\gamma_5(t)$ is substantially simplified when we substitute the above closure relations into the wave-packet expression of equation (17). To clarify this point, we suppose that the initial condition over $\psi(t, \boldsymbol{x})$ can be set in terms of the Fourier transform of the weight function

$$\varphi(\boldsymbol{p} - \boldsymbol{p}_i)w = \sum_{s=1,2} \{b(p_s)\mathcal{U}(p_s) + d^*(\tilde{p}_s)\mathcal{V}(\tilde{p}_s)\} \tag{31}$$

so that

$$\psi(0, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi(\boldsymbol{p} - \boldsymbol{p}_i) \exp[\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}]w, \tag{32}$$

where w is some fixed normalized spinor. By using the orthonormality properties of equation (30), we find [25]

$$b(p_s) = \varphi(\boldsymbol{p} - \boldsymbol{p}_i)\mathcal{U}^\dagger(p_s)w, \qquad d^*(\tilde{p}_s) = \varphi(\boldsymbol{p} - \boldsymbol{p}_i)\mathcal{V}^\dagger(\tilde{p}_s)w. \tag{33}$$

For *any* initial state $\psi(0, \boldsymbol{x})$ given by equation (32), the negative frequency solution coefficient $d^*(\tilde{p}_s)$ necessarily provides a non-null contribution to the time-evolving wave packet. This obliges us to take the complete set of Dirac equation solutions to construct a complete and correct wave-packet solution. Only if we consider the initial spinor w being a positive energy $(E_s(\boldsymbol{p}))$ and momentum $\boldsymbol{p}$ eigenstate, the contribution due to the negative frequency solutions $d^*(\tilde{p}_s)$ will become null and we will have a simple expression for the time evolution of any physical observable. By substituting the closure relations of equations (25) and (29) into the time-evolution expression for the above wave packet, equation (17) can thus be rewritten as

$$\begin{aligned} \psi(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi(\boldsymbol{p} - \boldsymbol{p}_i) \exp[\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}] \\ \times \sum_{s=1,2} \left\{ \left[\cos[E_s t] - \mathrm{i}\frac{H_s}{E_s} \sin[E_s t] \right] \left(\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right) \right\} w \end{aligned} \tag{34}$$

for the first case where E_s is given by equation (24) and $H_s = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \gamma_0 m_s$, or as

$$\psi(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi(\boldsymbol{p} - \boldsymbol{p}_i) \exp[\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}]$$
$$\times \sum_{s=1,2} \left\{ \left[\cos[E_s t] - \mathrm{i} \frac{H_0}{\varepsilon_0} \sin[E_s t] \right] \left(\frac{1 - (-1)^s \gamma_0 \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right) \right\} w \qquad (35)$$

for the second case where ε_0 is given by equation (28) and $H_0 = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \gamma_0 m$.

Once we have assumed that the neutrino electroweak interactions at the source and detector are (*left*) chiral ($\overline{\psi} \gamma^\mu (1 - \gamma^5) \psi W_\mu$) only the component with negative chirality contributes to the propagation. It was already demonstrated that, in vacuum, chiral oscillations can introduce very small modifications to the neutrino conversion formula [20, 24]. The probability that a neutrino produced as a negative chiral eigenstate be detected after a time t can be summarized by the expression

$$P(\boldsymbol{\nu}_{\alpha,L} \to \boldsymbol{\nu}_{\alpha,L}; t) = \int \mathrm{d}^3 \boldsymbol{x} \, \psi^\dagger(t, \boldsymbol{x}) \frac{1 - \gamma_5}{2} \psi(t, \boldsymbol{x}) = \frac{1}{2}(1 - \langle \gamma_5 \rangle(t)). \quad (36)$$

From this integral, it is readily seen that an initial -1 chiral mass-eigenstate will evolve with time changing its chirality. By assuming that the fermionic particle is created at time $t = 0$ as a -1 chiral eigenstate ($\gamma_5 w = -w$), in the case of $[\boldsymbol{\sigma} \cdot \boldsymbol{p}, \boldsymbol{\sigma} \cdot \boldsymbol{B}] = 0$ ($\boldsymbol{B}$ parallel to $\boldsymbol{p}$), we could write

$$\langle \gamma_5 \rangle(t) = \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi^2(\boldsymbol{p} - \boldsymbol{p}_i) w^\dagger \sum_{s=1,2} \left\{ \left[\gamma_5 \cos^2[E_s t] + \mathrm{i} \frac{[H_s, \gamma_5]}{2E_s} \sin[2E_s t] \right. \right.$$
$$\left. + \frac{H_s \gamma_5 H_s}{E_s^2} \sin^2[E_s t] \right] \left(\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right) \right\} w$$
$$= (-1) \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi^2(\boldsymbol{p} - \boldsymbol{p}_i) \sum_{s=1,2} \left\{ \left[\cos^2[E_s t] + \frac{p^2 - m_s^2}{E_s^2} \sin^2[E_s t] \right] \right.$$
$$\left. \times w^\dagger \left(\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right) w \right\}$$
$$= (-1) \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi^2(\boldsymbol{p} - \boldsymbol{p}_i) \sum_{s=1,2} \left\{ \left[\frac{p^2}{E_s^2} + \frac{m_s^2}{E_s^2} \cos[2E_s t] \right] w^\dagger \left(\frac{1 - (-1)^s \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}}}{2} \right) w \right\},$$
$$(37)$$

where we have used the wave-packet expression of equation (34) and, in the second passage, we have observed that

$$w^\dagger \gamma_5 w = -1, \qquad w^\dagger [H_s, \gamma_5] w = 0 \qquad \text{and} \qquad H_s \gamma_5 H_s = p^2 - m_s^2. \qquad (38)$$

The above expression can be reduced to a simpler one in the non-interacting case [24]. Due to a residual interaction with the external magnetic field $\boldsymbol{B}$ we could also observe chiral oscillations in the ultra-relativistic limit. However, from the phenomenological point of view, the coefficient of the oscillating term goes with $\frac{m_s^2}{E_s^2}$ which makes chiral oscillations become not relevant for ultra-relativistic neutrinos [15, 20]. As a *toy model* illustration, by assuming a highly peaked momentum distribution centred around a non-relativistic momentum $p_i \ll m_s$, where the wave-packet effects are practically ignored, the chiral conversion formula can be written as

$$P(\boldsymbol{\nu}_{\alpha,L} \to \boldsymbol{\nu}_{\alpha,R}; t) \approx \tfrac{1}{2}(1 - \cos[2mt] \cos[2|\boldsymbol{a}|t] - \sin[2mt] \sin[2|\boldsymbol{a}|t] w^\dagger \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}} w), \qquad (39)$$

where all the oscillating terms come from the interference between positive and negative frequency solutions which compose the wave packets. Turning back to the case where $\{\boldsymbol{\sigma} \cdot \boldsymbol{p}, \boldsymbol{\sigma} \cdot \boldsymbol{B}\} = 0$ ($\boldsymbol{B}$ orthogonal to $\boldsymbol{p}$), we could have a phenomenologically more interesting result. By following a similar procedure with the mathematical manipulations, we could write

$$
\begin{aligned}
\langle \gamma_5 \rangle(t) &= \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi^2(\boldsymbol{p} - \boldsymbol{p}_i) w^\dagger \left\{ \gamma_5 \cos[E_1 t] \cos[E_2 t] + \frac{H_0 \gamma_5 H_0}{\epsilon_0^2} \sin[E_1 t] \sin[E_2 t] \right. \\
&\quad \left. + \frac{\mathrm{i}}{2} [[H_0, \gamma_5] \sin[E_1 + E_2] - \{H_0, \gamma_5\} \gamma_0 \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}} \sin[E_1 - E_2]] \right\} w \\
&= (-1) \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi^2(\boldsymbol{p} - \boldsymbol{p}_i) \left\{ \cos[E_1 t] \cos[E_2 t] + \frac{\boldsymbol{p}^2 - m^2}{\epsilon_0^2} \sin[E_1 t] \sin[E_2 t] \right\} \\
&= (-1) \int \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \varphi^2(\boldsymbol{p} - \boldsymbol{p}_i) \left\{ \frac{\boldsymbol{p}^2}{\epsilon_0^2} \cos[2|\boldsymbol{a}|t] + \frac{m^2}{\epsilon_0^2} \cos[2\epsilon_0 t] \right\},
\end{aligned}
\tag{40}
$$

where we have used the wave-packet expression of equation (35) and, in addition to $w^\dagger \gamma_5 w = -1$, we have also observed that $\{H_0, \gamma_5\} = 2\gamma_5 \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}}$ and, subsequently, $w^\dagger \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}} \gamma_0 \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{a}} w = 0$. Now, in addition to the non-interacting oscillating term $\frac{m^2}{\epsilon_0^2} \cos[2\epsilon_0 t]$, which comes from the interference between positive and negative frequency solutions of the Dirac equation, we have an extra term which comes from the interference between equal sign frequencies and, for very large time scales, can substantially change the oscillating results. In this case, it is interesting to observe that the ultra-relativistic limit of equation (40) leads to the following expressions for the chiral conversion formulae,

$$
P(\boldsymbol{\nu}_{\alpha,L} \to \boldsymbol{\nu}_{\alpha,R}; t) \approx \tfrac{1}{2}(1 - \cos[2|\boldsymbol{a}|t]) \tag{41}
$$

and

$$
P(\boldsymbol{\nu}_{\alpha,L} \to \boldsymbol{\nu}_{\alpha,L}; t) \approx \tfrac{1}{2}(1 + \cos[2|\boldsymbol{a}|t]), \tag{42}
$$

which, differently from chiral oscillations in vacuum, can be phenomenologically relevant. Obviously, we are reproducing the consolidated results already attributed to neutrino spin-flipping [30] where, by taking the ultra-relativistic limit, the chirality quantum number can be approximated by the helicity quantum number, but now it was accurately derived from the complete formalism with Dirac spinors. We still remark that, in the standard treatment of vacuum neutrino oscillations, the use of scalar mass-eigenstate wave packets made up exclusively of the positive frequency plane-wave solutions is implicitly assumed. A satisfactory description of fermionic (spin one-half) particles requires the use of the Dirac equation as the evolution equation for the mass-eigenstates despite the standard oscillation formula giving the correct result when it is *properly* interpreted. We have observed in equations (40) and (37) that the spinorial form and the interference between positive and negative frequency components of the mass-eigenstate wave packets can introduce small modifications to the *standard* conversion formulae when they concern with *non-relativistic* neutrinos [15]. In the next step, by following the Dirac wave packet prescription, we intend to study the coupling of chiral oscillations with flavour conversion effects for neutrinos non-minimally interacting with an external magnetic field and, subsequently, verify the possibility of some phenomenological implications.

To summarize, we would have been dishonest if we had ignored the complete analysis of the general case comprising equations (10)–(13) where we had not yet assumed an arbitrary (simplified) spatial configuration for the magnetic field. Meanwhile, such a general case leads to the formal connection between quantum oscillation phenomena and a very different field. It concerns with the curious fact that those complete (general) expressions for propagating

wave packets do not satisfy the standard dispersion relations such as $E^2 = m^2 + p^2$ except the two particular cases where $E_s(p)^2 = m_s^2 + p^2$ for $p \times B = 0$ or $\epsilon_0^2 = m^2 + p^2$ for $p \cdot B = 0$. In principle, it could represent an inconvenient obstacle forbidding the extension of these restrictive cases to a general one. However, we believe that it can also represent a starting point in discussing dispersion relations which may be incorporated into frameworks encoding the breakdown (or the validity) of Lorentz invariance.

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